New Julia Sets of Ishikawa Iterates

Yashwant S Chauhan
Asst. Professor
Computer Science & Engg. Department
G. B. Pant Engg. College,
Pauri Garhwal

Rajeshri Rana
Asst. Professor
Applied Science and Humanities Department
G. B. Pant Engg. College,
Pauri Garhwal

Ashish Negi
Asst. Professor
Computer Science & Engg. Department
G. B. Pant Engg. College,
Pauri Garhwal

ABSTRACT
We investigate in this paper the dynamics and the method of generating fractal images for Ishikawa iteration procedure. The geometry of relative superior Julia sets are explored for Ishikawa iteration.

Keywords
Complex dynamics, relative superior Julia Set, Ishikawa Iteration.

1. INTRODUCTION
Many authors have presented the papers on several "orbit traps" rendering methods to create the artistic fractal images. An orbit trap is a bounded area in complex plane into which an orbiting point may fall. Motivated by this idea of "orbit traps", this paper introduces the different types of orbit traps for Ishikawa iteration procedure. It is well known that Julia sets of $z_{n+1} = az_n^2 + c$ are connected and bounded for *a* and *c*. So, we consider them as orbit traps and explore their relative fractal images.

2. PRELIMINARIES
Let $\{z_n : n = 1, 2, 3, 4.........\}$, denoted by $\{z_n\}$ be a sequence of complex numbers. Then, we say $\lim_{n\to\infty} z_n = \infty$ if for given M > 0, there exists N > 0, such that for all n > N, we must have $|z_n| > M$. Thus all the values of z_n, lies outside a circle of radius M, for sufficiently large values of n.

Let $Q(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{n-1} z^1 + a_n z^0 ; a_0 \neq 0$ be a polynomial of degree n, where $n \geq 2$. The coefficients are allowed to be complex numbers. In other words, it follows that $Q_c(z) = z^2 + c$.

Definition 2.1: Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, the Picard's orbit is defined as the set of iterates of a point x_0, that is; $O(f, x_0) = \{x_n ; x_n = f(x_{n-1}), n = 1, 2, 3.....\}$.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded, *i.e.* there exists M, such that $|Q^n(z)| \leq M$, for all n, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

Definition 2.2: The set of points K whose orbits are bounded under the iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

3. ISHIKAWA ITERATION FOR RELATIVE SUPERIOR JULIA SETS
Let X be a subset of real or complex numbers and $f : X \to X$. For $x_0 \in X$, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_0 = s'_0 f(x_0) + (1 - s'_0) x_0$$
$$y_1 = s'_1 f(x_1) + (1 - s'_1) x_1 \;.......$$
$$y_n = s'_n f(x_n) + (1 - s'_n) x_n$$

where $0 \leq s'_n \leq 1$ and s'_n is convergent to non zero number and

$$x_1 = s_0 f(y_0) + (1 - s_0) x_0$$
$$x_2 = s_1 f(y_1) + (1 - s_1) x_1 \;.......$$
$$x_n = s_{n-1} f(y_{n-1}) + (1 - s_{n-1}) x_{n-1}$$

where $0 \leq s_n \leq 1$ and s_n is convergent to non zero number[12].

Definition 3.1: The sequences x_n and y_n constructed above is called Ishikawa sequences of iteration or relative superior sequences of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$.

Notice that $RSO(x_0, s_n, s'_n, t)$ with s'_n=1 is $RSO(x_0, s_n, t)$

i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduces to $O(x_0, t)$. We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is Relative superior orbit. Now we define Julia set for function with respect to Ishikawa iterates. We call them as Relative Superior Julia sets.

Definition 3.2: The set of points SK whose orbits are bounded under Relative superior iteration of function Q(z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

We now define escape criterions for these sets.

3.1 Relative Superior Escape Criterions for Quadratics

The following theorem gives us an escape Criterions for function $Q_c = z^2 + c$ in respect to Ishikawa iteration procedure.

Theorem 3.1: Let's assume that $|z| \geq |c| > 2/s$; $|z| \geq |c| > 2/s'$, where $0 < s < 1$, $0 < s' < 1$ and *c* is a complex number. Define $z_1 = (1-s)z + sQ_c(z)$

$$\vdots$$

$$z_n = (1-s)z_{n-1} + sQ_c(z_{n-1})$$

where $Q_c(z)$ can be a quadratic, cubic or biquadratic polynomial in terms of s' and n = 2,3,4,... then $|z_n| \to \infty$, as $n \to \infty$.

Proof: Let's take $|Q_c(z)| = |(1-s')z + s'Q'_c(z)|$,

where $Q'_c(z) = z^2 + c$

$$= |s'z^2 + (1-s')z + s'c|$$

$$\geq |s'z^2 + (1-s')z| - |s'c|$$

$$\geq |z|(|s'z + (1-s')|) - s'|z| \quad (\because |z| \geq |c|)$$

$$\geq |z|(|s'z| - 1 + s') - s'|z|$$

$$= |z|(|s'z| - 1) \qquad \text{.... (1)}$$

Now since, $z_n = (1-s)z_{n-1} + sQ'_c(z)$

So, $|z_1| = |(1-s)z + sQ_c(z)|$ on substituting (1)

$$= |(1-s)z + s|z|(|s'z| - 1)|$$

$$= |z - sz + s|z|.|s'z| - s|z||$$

$$\geq (|z| + |sz|) + (s|z|.|s'z| - s|z|)$$

$$\geq |z| + |sz| + s|z|.|s'z| - s|z|$$

$\geq |z|(1 + ss'|z|)$, since $s|z| > 2$, so, $ss'|z| > 2$, there exists $\lambda > 0$, such that $ss'|z| - 1 > 1 + \lambda$

Consequently $|z_1| > (1+\lambda)|z|$

$$\vdots$$

$$|z_n| > (1+\lambda)^n |z|$$

Thus, the Ishikawa orbit of z, under the quadratic function tends to infinity. This completes the proof.

Corollary 3.1: Suppose that $|c| > 2/s$; $|c| > 2/s'$. Then, the relative superior orbit of Ishikawa $RSO(Q_c, 0, s, s')$ escapes to infinity.

In the proof of the theorem, we used the facts that $|z| \geq |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$. Hence, the following corollary is the refinement of the escape criterion discussed in the above theorem.

Corollary 3.2(Escape Criterion): Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$.

Corollary 3.3: Suppose that $|z_k| > \max\{|c|, 2/s, 2/s'\}$, for some $k \geq 0$. Then, $|z_{k+1}| > (1+\lambda)^n |z_k|$ and $|z_n| \to \infty$ as $n \to \infty$.

This corollary gives us an algorithm for computing the Relative Superior Julia sets of Q_c, for any c. Given any point $|z| \leq |c|$, we have computed the superior orbit of z. If for some n, $|z_n|$ lies outside the circle of radius $\max\{|c|, 2/s, 2/s'\}$, we guarantee that the orbit escapes. Hence, z is not in the Relative Superior Julia sets. On the other hand, if $|z_n|$ never exceeds this bound, then by definition of the Relative Superior Julia sets, denoted by RSK_c. We can make extensive use of this algorithm in the next section.

3.2 Relative Superior Escape Criterion for Cubic Polynomials:

First, we prove the following theorem for the function $Q_{a,b}(z) = z^3 + az + b$ with respect to the Ishikawa iteration procedure.

Theorem 3.2: Suppose $|z| > |b| > (|a| + 2/s)^{1/2}$, $|z| > |b| > (|a| + 2/s')^{1/2}$ exists, where $0 < s \leq 1; 0 < s' < 1$ and *a* and *b* are in complex plane .Define $z_1 = (1-s)z + sQ_{a,b}(z)$

$$\vdots$$

$z_n = (1-s)z_{n-1} + sQ_{a,b}(z_{n-1})$, n =2, 3……

where $Q_{a,b}(z)$ is the function of s', then $|z_n| \to \infty$ as $n \to \infty$.

Proof: Let's take $|Q_{a,b}(z)| = |(1-s')z + s'Q'_{a,b}(z)|$

$$= |(1-s')z + s'(z^3 + az + b)|$$

$$= |s'z^3 + s'az + z - s'z + bs'|$$

$\geq |s'z^3 + s'az + z - s'z| - |bs'|$

$\geq |z|(|s'z^2 + s'a + 1 - s'|) - s'|z| \qquad \because |z| \geq |b|$

$\geq |z|(|s'z^2 + as'| - |1 - s'|) - s'|z|$

$= |z|\{|s'z^2 + as'| - 1 + s' - s'\}$

$= |z|\{s'|z^2 + a| - 1\}$

$= s'|z|\{|z^2 + a| - 1/s'\}$

$\geq s'|z|\{|z|^2 - |a| - 1/s'\}$

$= s'|z|\{|z|^2 - (|a| + 1/s')\}$

Now since $|z_1| = |(1-s)z + sQ_{a,b}(z)|$

$= |(1-s)z + s.|z|.\{s'(|z^2 + a|) - 1\}|$

$= |z - sz + s|z|.s'(|z^2 + a|) - s|z||$

$\geq |z| + s|z| + \{s|z|.s'(|z^2 + a|) - s|z|\}$

$\geq |z| + s|z| + s.s'|z|(|z^2 + a|) - s|z|$

$\geq |z|\{1 + s.s'|z^2 + a|\}$

$= |z|s.s'(1/s.s' + |z^2 + a|)$

$\geq |z|s.s'(1/s.s' + |z^2| - |a|)$

$= |z|s.s'\{|z^2| - (|a| - 1/s.s')\}$

Since $|z| > (|a| + 2/s)^{1/2}$ and $|z| > (|a| + 2/s')^{1/2}$ exists and so $|z| > (|a| + 2/ss')^{1/2}$ follows .Therefore, $|z^2| - (|a| + 1/ss')^{1/2} > 1/ss'$ such that $ss'\{|z^2| - (|a| + 1/ss')\} > 1$. Hence, there exists $\gamma > 1$, such that $|z_1| > \gamma|z|$. Repeating this argument n time, we get $|z_n| > \gamma^n|z|$. Therefore, the Relative Superior orbit of z, under the cubic polynomial $Q_{a,b}(z)$, tends to infinity. This completes the proof.

Corollary 3.4: Suppose that $|b| > (|a| + 2/s)^{1/2}$ and $|b| > (|a| + 2/s')^{1/2}$ exists. Then, the Relative Superior orbit $RSO(Q_{a,b}, 0, s, s')$ escapes to infinity.

Corollary 3.5(Escape Criterion):
Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. Corollary 3.5 gives an escape criterion for cubic polynomials.

Corollary 3.6:
Assume that $|z_k| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ for some $k \geq 0$.Then $|z_{k+1}| > \gamma|z_k|$ and $|z_n| \to \infty$, as $n \to \infty$.From Corollary 3.6, we find an algorithm for computing the superior Julia sets of $Q_{a,b}(z)$, for any *a* and *b*.

3.3 A General Escape Criterion:

We will obtain a general escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 3.3: For general function $G_c(z) = z^n + c$, n = 1, 2, 3, 4 …where $0 < s \leq 1,\ 0 < s' < 1$ and *c* is the complex plane. Define $z_1 = (1-s)z + sG_c(z)$

$\vdots$

$z_n = (1-s)z_{n-1} + sG_c(z_{n-1})$

Thus, the general escape criterion is $\max\{|c|, (2/s)^{1/n+1}, (2/s')^{1/n+1}\}$.

Proof: We shall prove this theorem by induction:

For n = 1, we get $G_c(z) = z + c$. So, the escape criterion is |c|, which is obvious, *i.e.* $|z| > \max\{|c|, 0, 0\}$

For n = 2, we get $G_c(z) = z^2 + c$. So, the escape criterion is $|z| > \max\{|c|, 2/s, 2/s'\}$ (See Theorem 3.1)

For n = 3, we get $G_c(z) = z^3 + c$. So, the result follows from Theorem 3.2 with a = 0 and b = c, such that the escape criterion is $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$. Hence, the theorem is true for n = 1, 2, 3, 4…

Now, suppose that theorem is true for any n. Let $G_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2/s)^{1/n+1}$ as well as $|z| \geq |c| > (2/s')^{1/n+1}$ exists. Then,

$|G_n(z)| = |(1-s')z + s'G_c'(z)|$ where $G_c'(z) = z^{n+1} + c$

$= |z - s'z + s'(z^{n+1} + c)|$

$\geq |s'z^{n+1} - s'z + z| - s'|c|$

$\geq |z|(s'|z^n| - s' + 1| - s'|z| \qquad (\because |z| \geq |c|)$

$\geq |z|\{(s'|z^n| + |s'| - |1|\} - s'|z|$

$\geq |z|(s'|z^n| + s' - 1 - s')$

$\geq |z|(s'|z^n| - 1)$

Now, $|z_1| = |(1-s)z + sG_n(z)|$

$= |(1-s)z + s|z|(s'|z^n| - 1)|$

$= |z - sz + s|z|.s'|z^n| - s|z||$

$\geq (|z| - s|z|) + (ss'|z^{n+1}| - s|z|)$

$\geq (ss'|z^{n+1}| + |z|)$

$\geq |ss'z^{n+1}| - |z|$

$\geq |z|(ss'|z^n| - 1)$

Since $|z| > (2/s)^{1/n}$; $|z| > (2/s')^{1/n}$ and $|z| > (2/ss')^{1/n}$ .So $|z| > (2/ss')^{1/n}$, therefore $(ss'|z^n|-1) > 1$

Hence, for some $\lambda > 0$, we have $(ss'|z^n|-1) > 1+\lambda$.

Thus, $|z_1| > (1+\lambda)|z|$

$$\vdots$$

$$|z_n| = (1+\lambda)^n |z|$$

Therefore, the Ishikawa orbit of z under the iteration of $z^{n+1}+c$ tends to infinity. Hence $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ is the escape criterion. This proves the theorem.

Corollary 3.7: Suppose that $|c| > (2/s)^{1/n-1}$ and $|c| > (2/s')^{1/n-1}$ exists. Then, the Relative Superior orbit $RSO(G_c, 0, s, s')$ escapes to infinity.

Corollary3.8:

Assume that $|z_k| > \max\{|c|, (2/s)^{1/k-1}, (2/s')^{1/k-1}\}$ for some $k \geq 0$. Then $|z_{k+1}| > \gamma |z_k|$ and $|z_n| \to \infty$, as $n \to \infty$. This corollary gives an algorithm for computing the Relative Superior Julia sets for the functions of the form $G_c(z) = z^n + c$, n = 1, 2, 3, 4…

4. FIXED POINTS

4.1 Fixed points of quadratic polynomial

Table 1: Orbit of F(z) for **(z_0=-1.077560973 - 0.823761912i)** at **s=0.1 and s’=0.4**

Number of iteration *i*	**|F(z)|**	**Number of iteration *i***	**|F(z)|**
1	1.3564	16	1.716
2	1.0463	17	1.7164
3	0.94717	18	1.7166
4	1.0745	19	1.7167
5	1.3067	20	1.7167
6	1.5394	21	1.7167
7	1.7178	22	1.7167
8	1.8123	23	1.7167
9	1.8205	24	1.7167
10	1.7819	25	1.7167
11	1.7454	26	1.7167
12	1.7258	27	1.7167
13	1.718	28	1.7167
14	1.7158	29	1.7167
15	1.7156	30	1.7167

Here we observe that the value converges to a fixed point after 19 iterations

Figure 1. Orbit of *F*(*z*) for **(z_0=-1.077560973 -0.823761912i)** at **s=0.1 and s’=0.4**

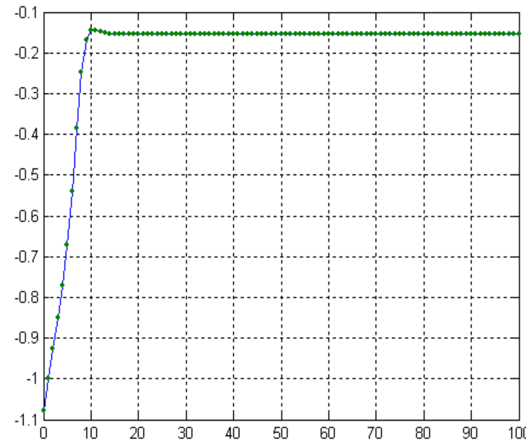


Table 2: Orbit of F(z) for **(z_0= -1.71 -0.24i)** at **s=0.3 and s’=0.4**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	1.7268	16	0.97304
2	1.3866	17	0.97304
3	1.2095	18	0.97304
4	1.132	19	0.97304
5	1.0602	20	0.97304
6	1.0051	21	0.97304
7	0.98257	22	0.97304
8	0.97564	23	0.97304
9	0.97371	24	0.97304
10	0.9732	25	0.97304
11	0.97308	26	0.97304
12	0.97305	27	0.97304
13	0.97304	28	0.97304
14	0.97304	29	0.97304
15	0.97304	30	0.97304

Here we observe that the value converges to a fixed point after 13 iterations

Figure 2. Orbit of *F*(*z*) for **(z_0= -1.71 -0.24i)** at **s=0.3 and s’=0.4**

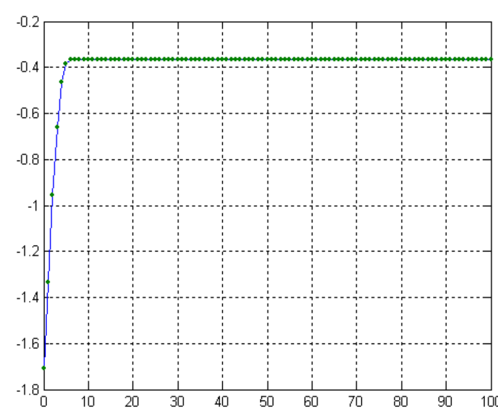
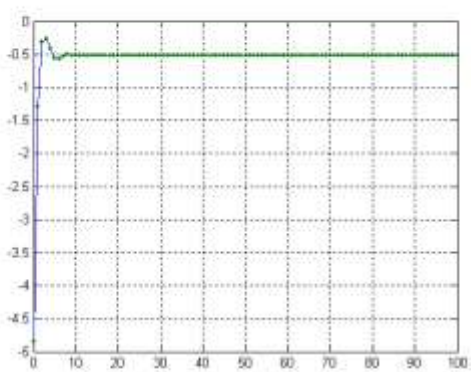


Table 3: Orbit of F(z) for **(z_0= -4.85 - 0.86i)** at **s=0.4 and s'=0.1**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	4.9257	16	0.69181
2	1.6058	17	0.69174
3	0.30906	18	0.69164
4	0.48244	19	0.69161
5	0.71099	20	0.69163
6	0.78288	21	0.69165
7	0.73382	22	0.69165
8	0.68229	23	0.69165
9	0.67418	24	0.69165
10	0.68627	25	0.69165
11	0.69467	26	0.69165
12	0.69487	27	0.69165
13	0.69224	28	0.69165
14	0.6909	29	0.69165
15	0.69109	30	0.69165

Here the value converges to a fixed point after 21 iterations

Figure 3. Orbit of *F*(z) for **(z_0= -4.85 - 0.86i)** at **s=0.4 and s'=0.1**



4.2 Fixed points of Cubic polynomial

Table 1: Orbit of F(z) for **(z_0= -0.082+0.056i)** at **s=0.6 and s'=0.4**

Number of iteration *i*	|F(z)|	Number of iteration *i*	|F(z)|
1	0.099298	16	0.4428
2	0.43343	17	0.4428
3	0.44234	18	0.4428
4	0.44277	19	0.4428
5	0.44279	20	0.4428
6	0.4428	21	0.4428
7	0.4428	22	0.4428
8	0.4428	23	0.4428
9	0.4428	24	0.4428
10	0.4428	25	0.4428
11	0.4428	26	0.4428
12	0.4428	27	0.4428
13	0.4428	28	0.4428
14	0.4428	29	0.4428
15	0.4428	30	0.4428

Here the value converges to a fixed point after 06 iterations

Figure 1 Orbit of F(z) for **(z_0= -0.082+0.056i)** at **s=0.6 and s'=0.4**

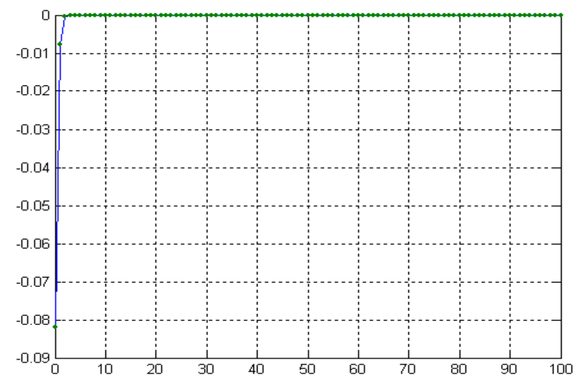


Table 2: Orbit of F(z) for **(z_0=-0.08+0.057i)** at **s=0.8 and s'=0.2**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
10	0.26107	26	0.26716
11	0.27154	27	0.26721
12	0.26406	28	0.26717
13	0.26942	29	0.2672
14	0.26558	30	0.26718
15	0.26833	31	0.26719
16	0.26637	32	0.26718
17	0.26777	33	0.26719
18	0.26677	34	0.26719
19	0.26749	35	0.26719
20	0.26697	36	0.26719
21	0.26734	37	0.26719
22	0.26708	38	0.26719
24	0.26727	39	0.26719
25	0.26713	40	0.26719

We skipped 09 iterations and after 33 iterations value converges

Figure 2. Orbit of *F*(z) for **(z_0=-0.08+0.057i)** at **s=0.8 and s'=0.2**

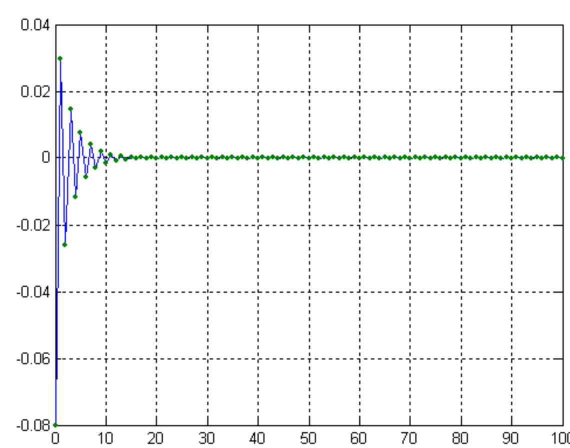
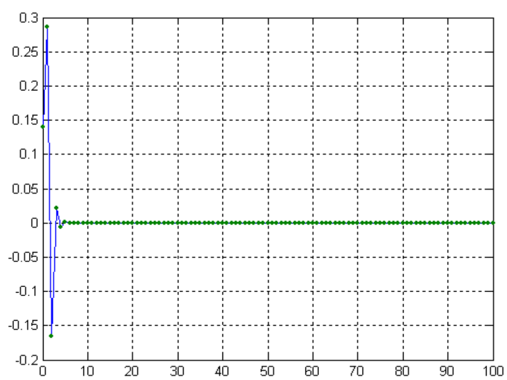


Table 3: Orbit of F(z) for **(z_0= 0.14+2.25i)** at **s=0.4 and s'=0.1**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	2.2544	16	0.47644
2	0.81868	17	0.47644
3	0.42429	18	0.47644
4	0.51067	19	0.47644
5	0.46793	20	0.47644
6	0.47848	21	0.47644
7	0.47594	22	0.47644
8	0.47655	23	0.47644
9	0.47641	24	0.47644
10	0.47644	25	0.47644
11	0.47643	26	0.47644
12	0.47644	27	0.47644
13	0.47644	28	0.47644
14	0.47644	29	0.47644
15	0.47644	30	0.47644

Here the value converges to a fixed point after 12 iterations

Figure 3. Orbit of F(z) for **(z_0= 0.14+2.25i)** at **s=0.4 and s'=0.1**



4.3 Fixed points of Bi-quadratic polynomial

Table 1: Orbit of F(z) for **(z_0= -0.046+0.165i)** at **s=0.6 and s'=0.4**

Number of iteration *i*	|F(z)|	Number of iteration *i*	|F(z)|
1	0.17129	16	0.62734
2	0.58304	17	0.62734
3	0.69194	18	0.62734
4	0.64174	19	0.62734
5	0.63002	20	0.62734
6	0.62746	21	0.62734
7	0.6272	22	0.62734
8	0.62728	23	0.62734
9	0.62732	24	0.62734
10	0.62733	25	0.62734
11	0.62734	26	0.62734
12	0.62734	27	0.62734
13	0.62734	28	0.62734
14	0.62734	29	0.62734
15	0.62734	30	0.62734

Here the value converges to a fixed point after 11 iterations

Figure 1 Orbit of F(z) for **(z_0= -0.046+0.165i)** at **s=0.6 and s'=0.4**

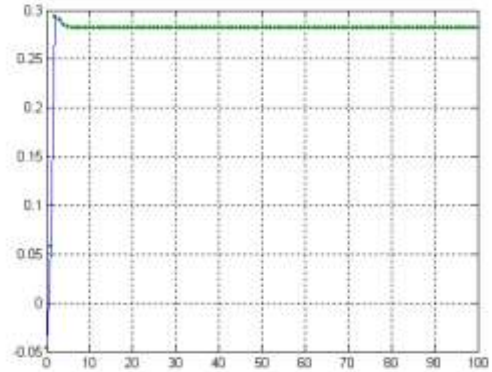


Table 2: Orbit of F(z) for **(z_0= 0.134+0.128i)** at **s=0.3 and s'=0.4**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
81	0.9200	96	0.9199
82	0.9196	97	0.9200
83	0.9197	98	0.9200
84	0.9201	99	0.9198
85	0.9201	100	0.9198
86	0.9198	101	0.9200
87	0.9197	102	0.9200
88	0.9200	103	0.9199
89	0.9201	104	0.9198
90	0.9199	105	0.9199
91	0.9197	106	0.9200
92	0.9199	107	0.9199
93	0.9200	108	0.9199
94	0.9199	109	0.9199
95	0.9198	110	0.9199

We skipped 81 iterations and after 107 iterations value converges

Figure 2 Orbit of F(z) for **(z_0= 0.134+0.128i)** at **s=0.3 and s'=0.4**

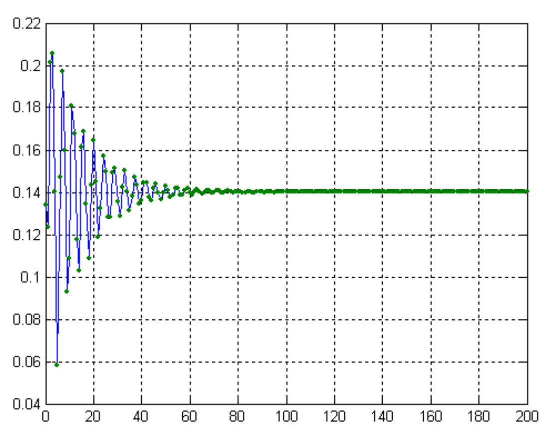
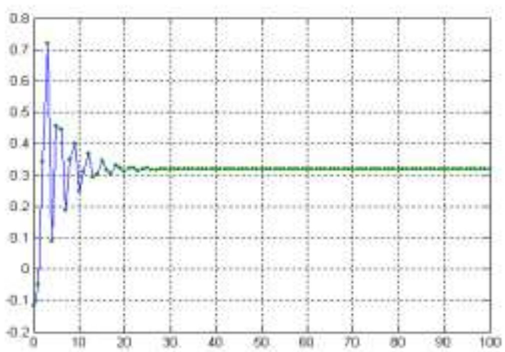


Table 3: Orbit of F(z) for **(z_0= -0.118+0.021i)** at **s=0.4 and s'=0.1**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
30	0.56705	45	0.56824
31	0.56825	46	0.56826
32	0.56908	47	0.5683
33	0.56782	48	0.56827
34	0.56801	49	0.56826
35	0.56872	50	0.56828
36	0.56819	51	0.56827
37	0.56803	52	0.56826
38	0.56847	53	0.56828
39	0.56831	54	0.56828
40	0.56812	55	0.56827
41	0.56833	56	0.56827
42	0.56834	57	0.56827
43	0.5682	58	0.56827
44	0.56827	59	0.56827

We skipped 29 iterations and after 55 iterations value converges

Figure 3. Orbit of F(z) for **(z_0= -0.118+0.021i)** at **s=0.4 and s'=0.1**



5. GENERATION OF RELATIVE SUPERIOR JULIA SETS:

We generated the Relative Superior Julia sets. We present here some beautiful filled Relative Superior Julia sets for quadratic, cubic and biquadratic function.

5.1 Relative Superior Julia sets for Quadratic:

Figure 1: Relative Superior Julia Set for s=s'=1, c = -1.38

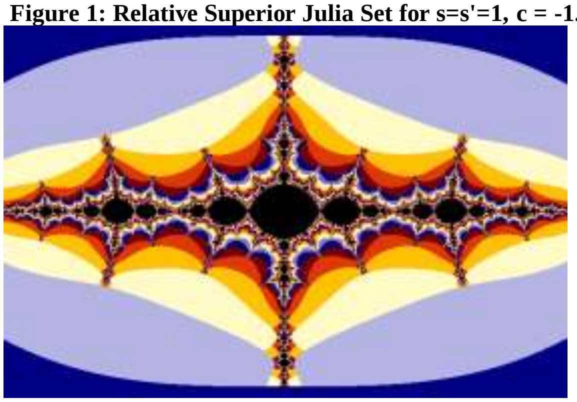


Figure 2: Relative Superior Julia Set for s=1, s'=0.3, c = 0.430+0.18*i*

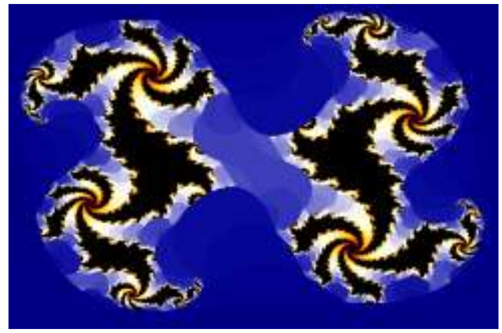


Figure 3: Relative Superior Julia Set for s=0.3, s'=1, c=-2.46

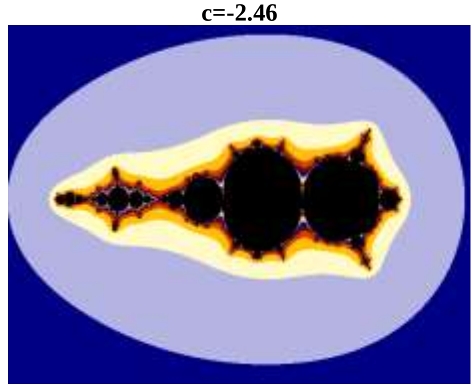


Figure 4: Relative Superior Julia Set for s=0.1, s'=0.4, c=-20.26+0.097*i*

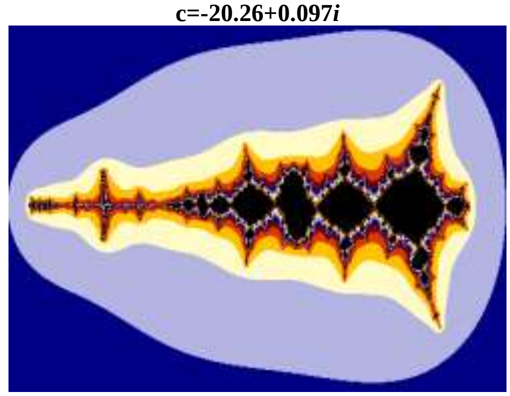
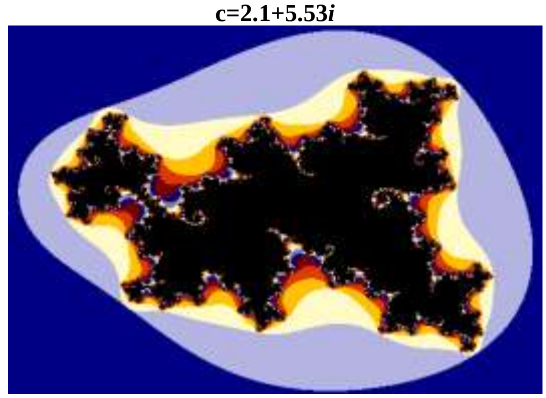


Figure 5: Relative Superior Julia Set for s=0.4, s'=0.1, c=2.1+5.53*i*



5.2 Relative Superior Julia sets for Cubic function:

Figure 1: Relative Superior Julia for s=s'=1, c=-0.2+1.1*i*

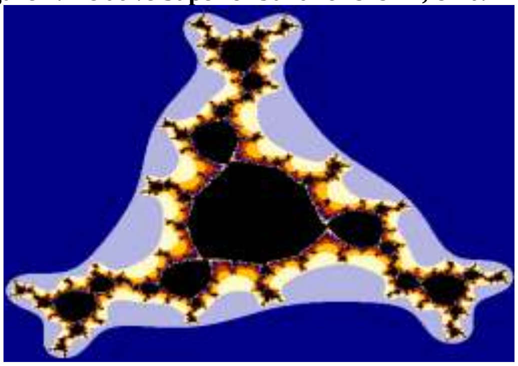


Figure 2: Relative Superior Julia Set for s=1, s'=0.5, c= -0.146+1.54*i*

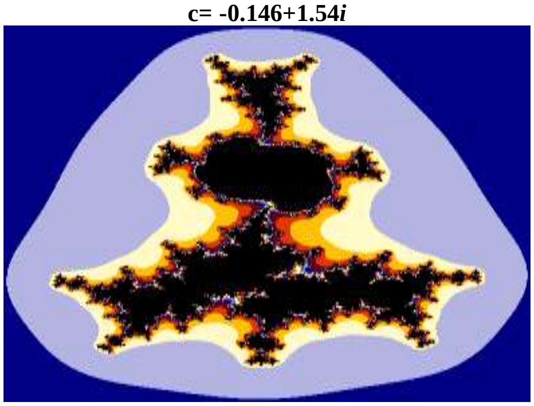


Figure 3: Relative Superior Julia Set for s=0.3, s'=1, c= -0.5+1.41*i*

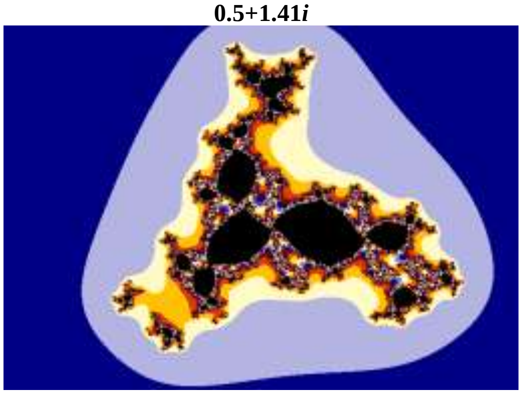


Figure 4: Relative Superior Julia Set for s=0.1, s'=0.4, c = -1.6+6.7i

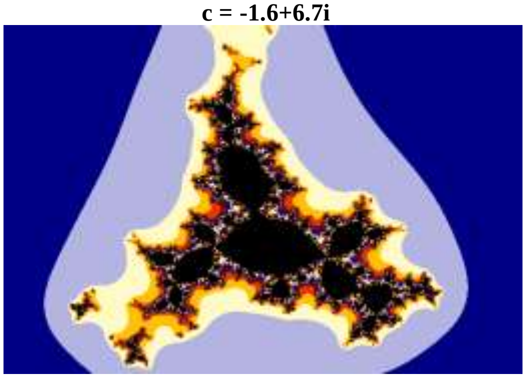
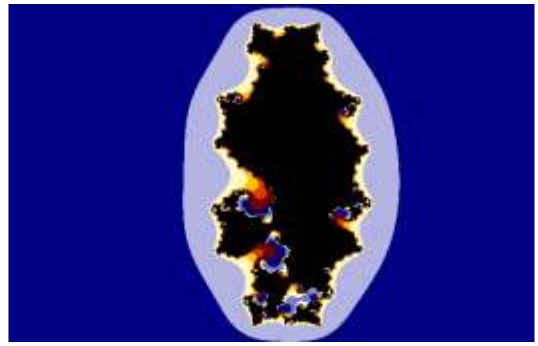


Figure 5: Relative Superior Julia Set for s=0.4, s'=0.1, c = -1+0.5i



5.3 Relative Superior Julia sets for Bi-quadratic function:

Figure 1: Relative Superior Julia for s=s'=1, c = 0.58 - 0.98*i*

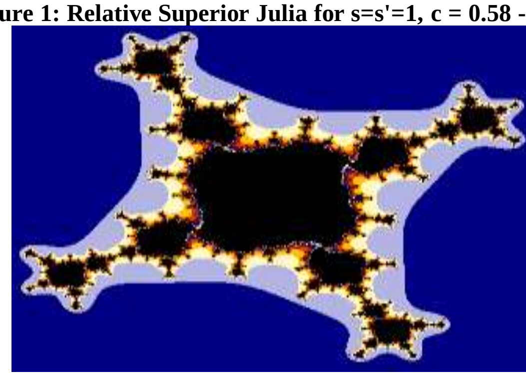


Figure 2: Relative Superior Julia Set for s=1, s'=0.5, c = -1.57

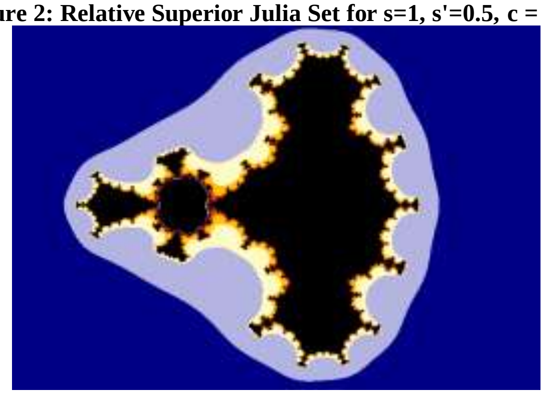


Figure 3: Relative Superior Julia Set for s=0.5, s'=1, c= -1.24

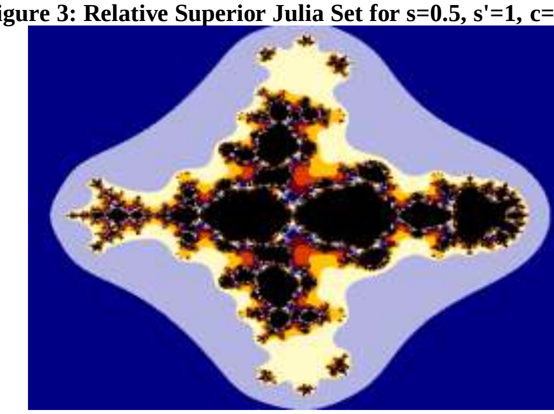


Figure 4: Relative Superior Julia Set for s=0.1, s'=0.4, c=2.6+0.0i

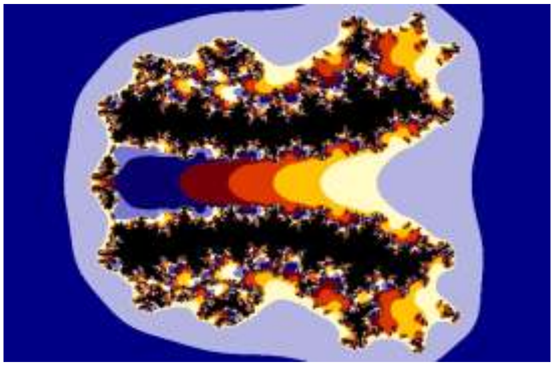
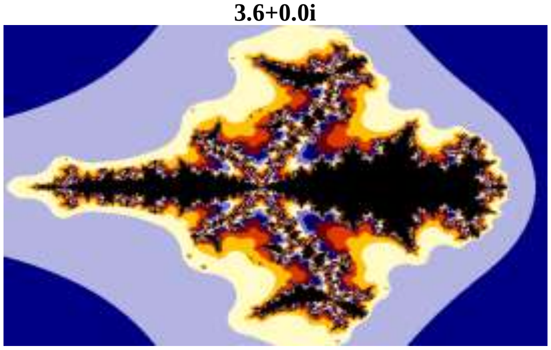


Figure 5: Relative Superior Julia Set for s=0.3, s'=0.4, c= -3.6+0.0i



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